# TOPOLOGICAL EQUIVALENCE OF COMPLEX POLYNOMIALS

ARNAUD BODIN AND MIHAI TIBĂR

Abstract. The following numerical control over the topological equivalence is proved: two complex polynomials in $n \neq 3$ variables and with isolated singularities are topologically equivalent if one deforms into the other by a continuous family of polynomial functions $f_s\colon \mathbb{C}^n \to \mathbb{C}$ with isolated singularities such that the degree, the number of vanishing cycles and the number of atypical values are constant in the family.

## 1. Introduction

Two polynomial functions $f, g : \mathbb{C}^n \to \mathbb{C}$ are said to be *topologically equivalent* if there exist homeomorphisms $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ and $\Psi : \mathbb{C} \to \mathbb{C}$ such that $\Psi \circ f = g \circ \Phi$. A challenging natural question is: *under what conditions this topological equivalence is controlled by numerical invariants?*

We shall assume that our polynomials have isolated critical points, and therefore finitely many. It appears that the topology of a polynomial function depends not only on critical points but also, since the function is non-proper, on the behaviour of its fibres in the neighbourhood of infinity. It is well-known (and goes back to Thom [21]) that a polynomial function has a finite set of atypical values $\mathcal{B} \subset \mathbb{C}$, i.e. values at which the function fails to be a locally trivial fibration. Even if the critical points of the polynomial are isolated, the homology of fibres may be very complicated, behaving as if highly non-isolated singularities occur at infinity. One has studied such kind of singularities at infinity in case they are in a certain sense isolated, e.g. [4], [19], [16], [20]. In this case the reduced homology of the general fibre $G$ is concentrated in dimension $n-1$ and certain numbers may be attached to singular points at infinity.

Coming back to topological equivalence: if our $f$ and $g$ are topologically equivalent then clearly their corresponding fibres (general or atypical) are homeomorphic. In particular the Euler characteristics of the general fibres of $f$ and $g$ and the numbers of atypical values of $f$ and $g$ coincide respectively. We prove the following numerical criterion for topological equivalence (see §2 for an example):

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**Theorem 1.** *Let $(f_s)_{s\in[0,1]}$ be a continuous family of complex polynomials with isolated singularities in the affine space and at infinity, in $n \neq 3$ variables. If the numbers $\chi(G(s))$, $\#\mathcal{B}(s)$ and $\deg f_s$ are independent of $s \in [0,1]$, then the polynomials $f_0$ and $f_1$ are topologically equivalent.*

In case of a smooth family of germs of holomorphic functions with isolated singularity $g_s : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, a famous result by Lê D.T. and C.P. Ramanujam [13] says that the constancy of the *local Milnor number* (equivalently, of the Euler characteristic of the general fibre in the local Milnor fibration [17]) implies that the hypersurface germs $g_0^{-1}(0)$ and $g_1^{-1}(0)$ have the same topological type whenever $n \neq 3$. J.G. Timourian [23] and H. King [9] showed moreover the topological triviality of the family of function germs over a small enough interval. The techniques which are by now available for proving the Lê-Ramanujam-Timourian-King theorem do not work beyond the case of isolated singularities. In other words, the topological equivalence problem is still unsolved for local non-isolated singularities.

The global setting poses new problems since one has to deal in the same time with several singular points and atypical values. Singularities at infinity introduce a new and essential difficulty since they are of a different type than the critical points of holomorphic germs. Some evidence for the crucial importance of singularities at infinity, even when assumed isolated, in understanding the behaviour of polynomials is the famous unsolved Jacobian Conjecture. One of the equivalent formulations of this conjecture, in two variables, is the following, see [14], [19]: if $f : \mathbb{C}^2 \to \mathbb{C}$ has no critical points but has isolated singularities at infinity then, for any polynomial $h : \mathbb{C}^2 \to \mathbb{C}$, the critical locus $Z(\mathrm{Jac}(f, h))$ is not empty.

Our approach consists in three main steps, which we briefly describe in the following.
*Step 1.* We show how the assumed invariance of numbers implies the rigidity of the singularities, especially of those at infinity. We use the following specific definition of *isolated singularities at infinity*, also employed in other papers (see e.g. Libgober [15] and Siersma-Tibăr [20]): the projective closure of any fibre of the polynomial and its slice by the hyperplane at infinity have isolated singularities. We explain in §3 how this condition enters in the proof of the key results on the semi-continuity of certain numbers, which in turn implies the rigidity of singularities. The class of polynomials with isolated singularities at infinity is large enough to include all polynomials of two variables with reduced fibres.

*Step 2.* Assuming the rigidity of some singularity, we prove a local topological triviality result. In case of a singular point in $\mathbb{C}^n$ this is Timourian's theorem, but we have to consider the new case of a singularity at infinity. To handle such a problem we first compactify all the polynomials of the family in the "same way", i.e. we consider the total space $\mathbb{X}$ of a family depending polynomially on the parameter: here we need the constancy of the degree. One cannot conclude by Thom-Mather's first Isotopy Lemma

since the natural stratification given by the trajectory of the singular point is not Whitney in general. Unlike the local case, in our setting the underlying space $\mathbb{X}$ turns out to be singular (essentially since the compactification has singularities at infinity). Our strategy is to revisit and modify the explicit local trivialisation given by Timourian's proof by taking into account the Whitney stratification of $\mathbb{X}$ (§5). The use of Timourian's proof is also responsible for the excepted case $n = 3$, due to an argument by Lê-Ramanujam which relies on the h-cobordism theorem, cf [13].

*Step 3.* Finally, we show how to patch together all the pieces (i.e. some open subsets of $\mathbb{C}^n$) where we have topological triviality, in order to obtain the global topological equivalence. The first named author used patching in [2] to prove topological equivalence in case there are no singularities at infinity and in case $n = 2$ with additional hypotheses and relying on results by L. Fourrier [7] which involve resolution of singularities. In our setting we have to deal with pieces coming from singularities at infinity and their patching is more delicate (see §6).

Let us remark that our theorem only requires the continuity of the family instead of the smoothness in [13], [23]. The reduction from a continuous family to a family depending polynomially on the parameter is made possible by a constructibility argument developed in §4. The constructibility also implies the finiteness of topological types of polynomials when fixing numerical invariants, see Remark 8. It is worth to point out that the finiteness does not hold for the equivalence up to diffeomorphisms, as already remarked by T. Fukuda [8]. For example, the family $f_s(x, y) = xy(x-y)(x-sy)$ provides infinitely many classes for this equivalence, because of the variation of the cross-ratio of the 4 lines.

## 2. Definitions and notations

We consider a one-parameter family of polynomials $f_s(x) = P(x, s)$, where $P : \mathbb{C}^n \times [0, 1] \to \mathbb{C}$ is polynomial in $s$ and such that $\deg f_s = d$, for all $s \in [0, 1]$.

We assume that the *affine singularities of* $f_s$ *are isolated*: $\dim \operatorname{Sing} f_s \leqslant 0$ for all $s$, where $\operatorname{Sing} f_s = \{x \in \mathbb{C}^n \mid \operatorname{grad} f_s(x) = 0\}$. The set of *affine critical values* of $f_s$ is a finite set and we denote it by $\mathcal{B}_{\mathit{aff}}(s) = \{t \in \mathbb{C} \mid \mu_t(s) > 0\}$, where $\mu_t(s)$ is the sum of the local Milnor numbers at the singular points of the fibre $f_s^{-1}(t)$; remark that we also have $\mathcal{B}_{\mathit{aff}}(s) = f_s(\operatorname{Sing} f_s)$. The *total Milnor number* is $\mu(s) = \sum_{t \in \mathcal{B}_{\mathit{aff}}} \mu_t(s)$. We also assume that, for all $s$, $f_s$ has isolated singularities at infinity in the following sense.

**Definition 2.** We say that a polynomial $f_s$ has *isolated singularities at infinity* if $\dim \operatorname{Sing} W(s) \leqslant 0$, where

$$W(s) = \left\{ [x] \in \mathbb{P}^{n-1} \mid \frac{\partial P_d}{\partial x_1} = \cdots = \frac{\partial P_d}{\partial x_n} = 0 \right\}$$

is an algebraic subset of the hyperplane at infinity $H^\infty$ of $\mathbb{P}^n$, which we identify to $\mathbb{P}^{n-1}$. Here $P_d$ denotes the homogeneous part of degree $d$ in variables $x = (x_1, \dots, x_n)$ of the polynomial $P(x, s)$. The condition $\operatorname{Sing} W(s) \leqslant 0$ is equivalent to the following: for all $t \in \mathbb{C}$, the singularities of $\overline{f_s^{-1}(t)}$ and of $\overline{f_s^{-1}(t)} \cap H^\infty$ are at most isolated.

In [20] one calls "$\mathcal{F}$ISI deformation of $f_0$" a family $P$ such that $f_s$ has isolated singularities at infinity and in the affine, for all $s$. The class of polynomials with isolated singularities at infinity is large enough to include all polynomial functions in two variables with reduced fibres. It is a (strict) subclass of polynomials having isolated singularities at infinity in the sense used by Broughton [4] or in the more general sense of [19].

We shall see in the following how one can precisely detect the singularities at infinity. We attach to the family $P$ the following hypersurface:

$$\mathbb{X} = \left\{([x : x_0], t, s) \in \mathbb{P}^n \times \mathbb{C} \times [0,1] \mid \tilde{P}(x, x_0, s) - t x_0^d = 0\right\},$$

where $\tilde{P}$ denotes the homogenisation of $P$ by the new variable $x_0$, considering $s$ as parameter. Let $\tau : \mathbb{X} \to \mathbb{C}$ be the projection to the $t$-coordinate. This extends the map $P$ to a proper one in the sense that $\mathbb{C}^n \times [0,1]$ is embedded into $\mathbb{X}$ and that the restriction of $\tau$ to $\mathbb{C}^n \times [0,1]$ is equal to $P$. Let $\sigma : \mathbb{X} \to [0,1]$ denote the projection to the $s$-coordinate. We shall use the notations $\mathbb{X}(s) = \sigma^{-1}(s) \cap \mathbb{X}$. Let $\mathbb{X}_t(s) = \mathbb{X}(s) \cap \tau^{-1}(t)$ be the projective closure in $\mathbb{P}^n$ of the affine hypersurface $f_s^{-1}(t)$. Note that $\mathbb{X}(s)$ is singular, in general with 1-dimensional singular locus, since $\operatorname{Sing} \mathbb{X}(s) = \Sigma(s) \times \mathbb{C}$, where:

$$\Sigma(s) = \left\{[x] \in \mathbb{P}^{n-1} \mid \frac{\partial P_d}{\partial x_1}(x, s) = \cdots = \frac{\partial P_d}{\partial x_n}(x, s) = P_{d-1}(x, s) = 0\right\}$$

and we have $\Sigma(s) \subset W(s)$, which implies that $\Sigma(s)$ is finite.

Let us fix some $s \in [0,1]$ and some $p \in \Sigma(s)$. For $t \in \mathbb{C}$, let $\mu_p(\mathbb{X}_t(s))$ denote the local Milnor number of the projective hypersurface $\mathbb{X}_t(s) \subset \mathbb{P}^n$ at the point $[p : 0]$. By [4] the number $\mu_p(\mathbb{X}_t(s))$ is constant for generic $t$, and we denote this value by $\mu_{p,gen}(s)$. We have $\mu_p(\mathbb{X}_t(s)) > \mu_{p,gen}(s)$ for a finite number of values of $t$. The *Milnor-Lê number* at the point $([p : 0], t) \in \mathbb{X}(s)$ is defined as the jump $\lambda_{p,t}(s) := \mu_p(\mathbb{X}_t(s)) - \mu_{p,gen}(s)$. We say that the point $([p : 0], t)$ is a *singularity at infinity* of $f_s$ if $\lambda_{p,t}(s) > 0$. Let $\lambda_t(s) = \sum_{p \in \Sigma(s)} \lambda_{p,t}(s)$. The set of *critical values at infinity* of the polynomial $f_s$ is defined as:

$$\mathcal{B}_{inf}(s) = \{t \in \mathbb{C} \mid \lambda_t(s) > 0\}.$$

Finally, the *Milnor-Lê number at infinity* of $f_s$ is defined as:

$$\lambda(s) = \sum_{t \in \mathcal{B}_{inf}(s)} \lambda_t(s).$$

For such a polynomial, the *set of atypical values*, or the *bifurcation set*, is:

$$\mathcal{B}(s) = \mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_{\mathit{inf}}(s).$$

It is known that $f_s : f_s^{-1}(\mathbb{C}\setminus\mathcal{B}(s)) \to \mathbb{C}\setminus\mathcal{B}(s)$ is a locally trivial fibration [4]. After [19], for $t \in \mathbb{C}$ the fibre $f_s^{-1}(t)$ is homotopic to a wedge of spheres of real dimension $n-1$ and the number of these spheres is $\mu(s)+\lambda(s)-\mu_t(s)-\lambda_t(s)$. In particular, for the Euler characteristic of the general fibre $G(s)$ of $f_s$ one has:

$$\chi(G(s)) = 1 + (-1)^{n-1}(\mu(s) + \lambda(s)).$$

**Example 3.** Let $f_s(x,y,z,w) = x^2y^2 + z^2 + w^2 + xy + (1+s)x^2 + x$. For $s \in \mathbb{C} \setminus \{-2,-1\}$ we have $\mathcal{B}(s) = \left\{0, -\frac{1}{4}, -\frac{1}{4}\frac{s+2}{s+1}\right\}$, $\mu(s) = 2$ and $\lambda(s) = 1$. It follows that $\chi(G(s)) = 1 - \mu(s) - \lambda(s) = -2$ and that $\#\mathcal{B}(s) = 3$. For the two excepted polynomials $f_{-1}$ and $f_{-2}$ we have $\#\mathcal{B} = 2$. Then, by Theorem 1, $f_0$ is topologically equivalent to $f_s$ if and only if $s \in \mathbb{C} \setminus \{-2,-1\}$.

Let $\mathbb{X}^\infty(s)$ denote the part "at infinity" $\mathbb{X}(s) \cap \{x_0 = 0\}$. We shall use in §5 the Whitney stratification of the space $\mathbb{X}(s)$ with the following strata (see [19], [20]): $\mathbb{X}(s) \setminus \mathbb{X}^\infty(s)$, $\mathbb{X}^\infty(s) \setminus \operatorname{Sing}\mathbb{X}(s)$, the complement in $\operatorname{Sing}\mathbb{X}(s)$ of the singularities at infinity and the finite set of singular points at infinity. We also recall that the restriction $\tau : \mathbb{X}(s) \to \mathbb{C}$ is transversal to all the strata of $\mathbb{X}(s)$ except at the singular points at infinity.

## 3. Rigidity of singularities in families of polynomials

Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials with constant degree $d$, such that the coefficients of $f_s$ are polynomial functions of $s \in [0,1]$. We also suppose that for all $s \in [0,1]$, $f_s$ has isolated singularities in the affine space and at infinity (in the sense of Definition 2). Under these conditions, we may prove the following rigidity result:

**Proposition 4.** *If the pair of numbers $(\mu(s)+\lambda(s), \#\mathcal{B}(s))$ is independent of $s$ in some neighbourhood of $0$, then the 5-uple $(\mu(s), \#\mathcal{B}_{\mathit{aff}}(s), \lambda(s), \#\mathcal{B}_{\mathit{inf}}(s), \#\mathcal{B}(s))$ is independent of $s$ too. Moreover there is no collision of points $p(s) \in \Sigma(s)$ as $s \to 0$, and in particular $\#\Sigma(s)$ and $\mu_{p,gen}(s)$ are constant.*

*Proof.* **Step 1.** We claim that the multivalued map $s \mapsto \mathcal{B}(s)$ is continuous. If not the case, then there is some value of $\mathcal{B}(s)$ which disappears as $s \to 0$. To compensate this, since $\#\mathcal{B}(s)$ is constant, there must be a value which appears in $\mathcal{B}(0)$. By the local constancy of the total Minor number, affine singularities cannot appear from nothing, therefore the new critical value should be in $\mathcal{B}_{\mathit{inf}}(0)$. More precisely, there is a singular point at infinity $(p,t)$ of $f_0$ (thus where the local $\lambda$ is positive) such that, for $s \neq 0$, there is no singular point of $f_s$, either in affine space or at infinity, which tends to $(p,t)$ as $s \to 0$. But this situation has been shown to be impossible in [2, Lemma 20]. Briefly, the argument goes as follows: Let $(p, c(0))$ be a singularity at infinity of $f_0$ and let $h_{s,t} : \mathbb{C}^n \to \mathbb{C}$ be the localisation at $p$

of the map $\tilde{P}(x, x_0, s) - tx_0^d$. Then from the local conservation of the total Milnor number of $h_{0,c(0)}$ and the dimension of the critical locus of the family $h_{s,t}$ one draws a contradiction. The claim is proved.

Let us remark that our proof also implies that the finite set $\mathcal{B}(s) \subset \mathbb{C}$ is contained in some disk of radius independent of $s$ and that there is no collision of points of $\mathcal{B}(s)$ as $s \to 0$.

**Step 2.** We prove that there is no collision of points $p(s) \in \Sigma(s)$ as $s \to 0$ and that $\#\Sigma(s)$ and $\mu_{p,gen}(s)$ are constant. We pick up and fix a value $t \in \mathbb{C}$ such that $t \notin \mathcal{B}(s)$, for all $s$ near 0. Then we have a one parameter family of general fibres $f_s^{-1}(t)$, where $s$ varies in a neighbourhood of 0. The corresponding compactified hypersurfaces $\mathbb{X}_t(s)$ have isolated singularities at their intersections with the hyperplane at infinity $H^\infty$.

Let $\mu_p^\infty(s)$ denote the Milnor number of the hyperplane slice $\mathbb{X}_t(s) \cap H^\infty$ at some $p \in W(s)$, and note that this does not depend on $t$, for some fixed $s$. We use the following formula (see [20, 2.4] for the proof and references):

$$\mu(s) + \lambda(s) = (d-1)^n - \sum_{p \in \Sigma(s)} \mu_{p,gen}(s) - \sum_{p \in W(s)} \mu_p^\infty(s). \tag{3.1}$$

Since $\mu(s)+\lambda(s)$ is constant and since the local upper semi-continuity of Milnor numbers, we have that both sums $\sum_{p\in\Sigma(s)} \mu_{p,gen}(s)$ and $\sum_{p\in W(s)} \mu_p^\infty(s)$ are constant hence locally constant. The non-splitting principle (see [10] or [12], [1]) applied to our family of hypersurface multigerms tells that each $\mu_{p,gen}(s)$ has to be constant. This means that there cannot be collision of points of $\Sigma(s)$.

**Step 3.** We claim that $\mu(s)$ is constant. If not the case, then we may suppose that $\mu(0) < \mu(s)$, for $s$ close to 0, since $\mu(s)$ is lower semi-continuous (see [4]). Then by also using Step 1, there exists $c(s) \in \mathcal{B}_{\mathit{aff}}(s)$, such that: $c(s) \to c(0) \in \mathbb{C}$ as $s \to 0$. By Step 1, there is no other value except $c(s) \in \mathcal{B}(s)$ which tends to $c(0)$. We therefore have a family of hypersurfaces $\mathbb{X}_{c(s)}(s)$ with isolated singularities $q_j(s) \in f_s^{-1}(c(s))$ that tend to the singularity at $(p, 0) \in \Sigma(0) \subset \mathbb{X}_{c(0)}(0)$. By Step 2 and the (upper) semi-continuity of the local Milnor numbers we have:

$$\mu_p(\mathbb{X}_{c(0)}(0)) \geqslant \mu_p(\mathbb{X}_{c(s)}(s)) + \sum_j \mu_{q_j(s)}(\mathbb{X}_{c(s)}(s)). \tag{3.2}$$

By definition, $\mu_p(\mathbb{X}_{c(s)}(s)) = \lambda_p(s) + \mu_{p,gen}(s)$ for any $s$, and by Step 2, $\mu_{p,gen}(s)$ is independent of $s$. It follows that:

$$\lambda_{p,c(0)}(0) \geqslant \lambda_{p,c(s)}(s) + \sum_j \mu_{q_j(s)}(\mathbb{X}_{c(s)}(s)), \tag{3.3}$$

which actually expresses the balance at any collision of singularities at some point at infinity. This shows that in such collisions the "total quantity of singularity", i.e. the local $\mu + \lambda$, is upper semi-continuous. On the other hand, the global $\mu + \lambda$ is assumed constant, by our hypothesis. This implies

that the local $\mu+\lambda$ is constant too. Therefore in (3.3) we must have equality and consequently (3.2) is an equality too.

We may now conclude by applying the non-splitting principle, similarly as in Step 2, to yield a contradiction.

**Step 4.** Since by Step 3 there is no loss of $\mu$, the multi-valued function $s \mapsto \mathcal{B}_{\mathit{aff}}(s)$ is continuous. Steps 1 and 3 show that $s \mapsto \mathcal{B}_{\mathit{inf}}(s)$ is continuous too. Together with $\#(\mathcal{B}_{\mathit{aff}}(s) \cup \mathcal{B}_{\mathit{inf}}(s)) = \mathrm{cst}$, this implies that $\#\mathcal{B}_{\mathit{aff}}(s) = \mathrm{cst}$ and $\#\mathcal{B}_{\mathit{inf}}(s) = \mathrm{cst}$. □

## 4. Constructibility via numerical invariants

Let $\mathcal{P}_{\leqslant d}$ be the vector space of all polynomials in $n$ complex variables of degree at most $d$. We consider here the subset $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B}) \subset \mathcal{P}_{\leqslant d}$ of polynomials of degree $d$ with fixed $\mu+\lambda$ and fixed $\#\mathcal{B}$.

Recall that a *locally closed set* is the intersection of a Zariski closed set with a Zariski open set; a *constructible set* is a finite union of locally closed sets.

**Proposition 5.** *$\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$ is a constructible set.*

*Proof.* The set $\mathcal{P}_d$ of polynomials of degree $d$ is a constructible set in the vector space $\mathcal{P}_{\leqslant d}$. Let us first prove that "isolated singularities at infinity" yields a constructible set. A polynomial $f$ has isolated singularities at infinity if and only if $W := W(f)$ has dimension 0 or is void. Let $S = \{(x, f) \in \mathbb{P}^n \times \mathcal{P}_d \mid f \in \mathcal{P}_d,\ x \in W(f)\}$ and let $\pi : S \to \mathcal{P}_d$ be the projection on the second factor. Since this is an algebraic map, by Chevalley's Theorem (e.g. [6, §14.3]) the set $\{f \in \mathcal{P}_d \mid \dim \pi^{-1}(f) \leqslant 0\}$ is constructible and this is exactly the set of polynomials with isolated singularities at infinity.

Next, we prove that fixing each integer $\mu$, $\#\mathcal{B}_{\mathit{aff}}$, $\lambda$, $\#\mathcal{B}_{\mathit{inf}}$, $\#\mathcal{B}$ yields a constructible set. The main reason is the semi-continuity of the Milnor number (upper in the local case, lower in the affine case), see e.g. Broughton [4, Prop. 2.3]. Broughton proved that the set of polynomials with a given $\mu < \infty$ is constructible. As the inverse image of a constructible set by an algebraic map, the set of polynomials with Milnor number $\mu$ and bifurcation set such that $\#\mathcal{B}_{\mathit{aff}} = k$ is a constructible set.

Let $\mathcal{P}_d(\mu, \#\Sigma)$ be the set of polynomials of degree $d$, with a given $\mu$, with isolated singularities at infinity and a given $\#\Sigma$. Notice that $\#\Sigma$ is finite because $\Sigma \subset W$ and is bounded for fixed $d$. Since $\Sigma$ depends algebraically on $f$, we have that $\mathcal{P}_d(\mu, \#\Sigma)$ is a constructible set. Now the local Milnor number $\mu_p$ is an upper semi-continuous function, so fixing $\lambda_p$ as the difference of two local Milnor numbers (see §2) provides a constructible set. By doing this for all the critical points at infinity we get that fixing $\lambda = \sum_p \lambda_p$ yields a constructible condition. The arguments for the conditions $\#\mathcal{B}_{\mathit{inf}}$ and $\#\mathcal{B}$ (which are numbers of points of two algebraic sets in $\mathbb{C}$) are similar to the one for $\#\mathcal{B}_{\mathit{aff}}$.

The just proved constructibility of $\mathcal{P}_d(\mu, \#\mathcal{B}_{\mathit{aff}}, \lambda, \#\mathcal{B}_{\mathit{inf}}, \#\mathcal{B})$ implies, by taking a finite union, the constructibility of $\mathcal{P}_d(\mu + \lambda, \#\mathcal{B})$. □

**Definition 6.** We say that a finite set $\Omega(s)$ of points in $\mathbb{C}^k$, for some $k$, depending on a real parameter $s$, is an *algebraic braid* if $\Omega = \cup_s \Omega(s) \times \{s\}$ is a real algebraic sub-variety of $\mathbb{C}^k \times [0,1]$, the multi-valued function $s \mapsto \Omega(s)$ is continuous and $\#\Omega(s) = \mathrm{cst}$.

We may now reformulate and extend Proposition 4 as follows.

**Proposition 7.** *Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials with isolated singularities in the affine space and at infinity, whose coefficients are polynomial functions of $s$. Suppose that the numbers $\mu(s)+\lambda(s)$, $\#\mathcal{B}(s)$ and $\deg f_s$ are independent of $s \in [0,1]$. Then:*

(1) *$\Sigma(s)$, $\mathcal{B}_{\mathit{aff}}(s)$, $\mathcal{B}_{\mathit{inf}}(s)$ and $\mathcal{B}(s)$ are algebraic braids;*
(2) *for any continuous function $s \mapsto p(s) \in \Sigma(s)$ we have $\mu_{p(s),gen} = \mathrm{cst}$;*
(3) *for any continuous function $s \mapsto c(s) \in \mathcal{B}_{\mathit{inf}}(s)$ we have $\lambda_{p(s),c(s)} = \mathrm{cst}$;*
(4) *for any continuous function $s \mapsto c(s) \in \mathcal{B}_{\mathit{aff}}(s)$ we have $\mu_{c(s)} = \mathrm{cst}$ and moreover, the local $\mu$'s of the fibre $f_s^{-1}(c(s))$ are constant.*

*Proof.* (1) For $\Sigma(s)$, it follows from the algebraicity of the definition of $\Sigma$ and from Step 2 of Proposition 4. It is well-known that affine critical values of polynomials are algebraic functions of the coefficients. Together with Proposition 4, this proves that $\mathcal{B}_{\mathit{aff}}(s)$ is an algebraic braid.

Similarly $\cup_s \mathcal{B}_{\mathit{inf}}(s) \times \{s\}$ is the image by a finite map of an algebraic set, and together with Step 4 of Proposition 4, this proves that $\mathcal{B}_{\mathit{inf}}(s)$ is an algebraic braid.

Next, (2) is Step 2 of Proposition 4 and (3) is a consequence of Step 3. Lastly, observe that (4) is a well-known property of local isolated hypersurface singularities and follows from (1) and the local non-splitting principle. □

**Remark 8.** Theorem 1 has the following interpretation: to a connected component of $\mathcal{P}_d(\mu + \lambda, \#\mathcal{B})$ one associates a unique topological type. (It should be noticed that two different connected components of $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$ may have the same topological type, see [3] for an example.) It follows that there is a finite number of topological types of complex polynomials of fixed degree and with isolated singularities in the affine space and at infinity. This may be related to the finiteness of topological equivalence classes in $\mathcal{P}_{\leqslant d}$, conjectured by René Thom and proved by T. Fukuda [8].

## 5. Local triviality at infinity

The aim of this section is to prove a topological triviality statement for a singularity at infinity. Our situation is new since it concerns a family of couples space-function varying with the parameter $s$ and where the space is singular. The proof actually relies on Timourian's proof [23] for germs of

holomorphic functions on $\mathbb{C}^n$. Therefore we shall point out where and how this proof needs to be modified, since we have to plug-in a singular stratified space (i.e. the germ of $\mathbb{X}(s)$ at a singularity at infinity) instead of the germ $(\mathbb{C}^n, 0)$ in Timourian's proof.

As before, let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials of degree $d$ and let $(p, c)$ be a singularity at infinity of $f_0$. Let $g_s : \mathbb{X}(s) \to \mathbb{C}$ be the localisation at $(p(s), c(s))$ of the map $\tau_{|\mathbb{X}(s)}$. We denote by $B_\varepsilon \subset \mathbb{C}^n \times \mathbb{C}$ the closed $2n + 2$-ball of radius $\varepsilon$ centred at $(p, c)$, such that $B_\varepsilon \cap \mathbb{X}(0)$ is a Milnor ball for $g_0$. We choose $0 < \eta \ll \varepsilon$ such that we get a Milnor tube $T_0 = B_\varepsilon \cap \mathbb{X}(0) \cap g_0^{-1}(D_\eta(c))$. Then, for all $t \in D_\eta(c)$, $g_0^{-1}(t)$ intersects transversally $S_\varepsilon = \partial B_\varepsilon$. We recall from [19] that $g_0 \colon T_0 \backslash g_0^{-1}(c) \to D_\eta(c) \backslash \{c\}$ is a locally trivial fibration whenever $\lambda_{p,c}(0) > 0$ and $g_0 \colon T_0 \to D_\eta(c)$ is a trivial fibration whenever $\lambda_{p,c}(0) = 0$.

According to Proposition 7(1), by an analytic change of coordinates, we may assume that $(p(s), c(s)) = (p, c)$ for all $s \in [0, u]$, for some small enough $u > 0$. We set $T_s = B_\varepsilon \cap \mathbb{X}(s) \cap g_s^{-1}(D_\eta(c))$ and notice that $B_\varepsilon$ does not necessarily define a Milnor ball for $g_s$ whenever $s \neq 0$. For some $u > 0$, let $T = \bigcup_{s\in[0,u]} T_s \times \{s\}$, and let $G : T \to \mathbb{C} \times [0, u]$ be defined by $G(z, s) = (g_s(z), s)$.

The homeomorphisms between the tubes that we consider here are all *stratified*, sending strata to corresponding strata. The stratification of some tube $T_s$ has by definition three strata: $\{T_s \setminus (\{p\} \times D_\eta(c)), \{p\} \times D_\eta(c) \setminus (p, c), (p, c)\}$.

**Theorem 9.** *Let $f_s(x) = P(x, s)$ be a one-parameter polynomial family of polynomial functions of constant degree, such that the numbers $\mu(s) + \lambda(s)$ and $\#\mathcal{B}(s)$ are independent of $s$. If $n \neq 3$, then there exists $u > 0$ and a homeomorphism $\alpha$ such that the following diagram commutes:*

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "G"'] & T_0 \times [0,u] \arrow[d, "g_0\times\mathrm{id}"] \\ D_\eta(c) \times [0,u] \arrow[r, "\mathrm{id}"'] & D_\eta(c) \times [0,u], \end{tikzcd}$$

*and such that $\alpha$ sends the strata of every $T_s$ to the corresponding strata of $T_0$.*

*Proof.* Our point $(p, c) \in \Sigma(0) \times \mathbb{C}$ is such that $\lambda_{p,c}(0) > 0$. We cannot apply directly Timourian's result for the family $g_s$ because each function $g_s$ is defined on a *singular* space germ $(\mathbb{X}(s), (p, c))$, but we can adapt it to our situation. To do this we recall the main lines of this proof and show how to take into account the singularities via the stratification of $T_s$.

Remark first that $(p, c)$ is the only singularity of $g_s$ in $T_s$, by the rigidity result Proposition 7. We use the notion of $\varepsilon$-homeomorphism, meaning a homeomorphism which moves every point within a distance no more than $\varepsilon > 0$.

Theorem 9 will follow from Lemma 10 below, once we have proved that the assumptions of this lemma are fulfilled in our new setting. This is a simplified statement of Timourian's Lemma 3 in [23], its proof is purely topological and needs no change.

**Lemma 10.** ([23, Lemma 3]) *Assume that:*

(1) *The space of stratified homeomorphisms of $T_0$ into itself, preserving the fibres of $g_0$, is locally contractible.*
(2) *For any $\varepsilon > 0$ there exists $u > 0$ small enough such that for any $s, s' \in [0, u]$ there is a stratified $\varepsilon$-homeomorphism $h : T_s \to T_{s'}$ with $g_s = g_{s'} \circ h$.*

*Then there exists a homeomorphism $\alpha$ as in Theorem 9.*

The assumptions (1) and (2) correspond, respectively, to Lemma 1 and Lemma 2 of Timourian's paper [23]. The remaining proof is therefore devoted to showing why the assumptions (1) and (2) are true in our setting.

Condition (1) can be proved as follows. It is well-known that analytic sets have local conical structure [5]. Notice that the stratification $\{T_0 \setminus (\{p\} \times D_\eta(c)), \{p\} \times D_\eta(c) \setminus (p, c), (p, c)\}$ of $T_0$ is a Whitney stratification (but that this is not necessarily true for tubes $T_s$ with $s \neq 0$). Timourian shows how to construct a vector field on $T_0$ such that all integral curves end at the central point $(p, c)$. Moreover, this vector field can be chosen such that to respect the Whitney strata. This is the only new requirement that we need to plug in. The rest of Timourian's argument remains unchanged once we have got the vector field, and we give only its main lines in the following. This vector field is used to define a continuous family $h_t$ of homeomorphisms such that $g_0 = g_0 \circ h_t$, which deforms a homeomorphism $h_1 = h$ of $T_0$ which is the identity at the boundary $\partial T_0$, to a homeomorphism $h_0$ which is the identity within a neighbourhood of $(\partial B_\varepsilon \cap T_0) \cup g_0^{-1}(c) \setminus (p, c)$. Next, by using the contracting vector field, one constructs an isotopy of $h_0$ to the identity, preserving the fibres of $g_0$. To complete the proof, Timourian shows how to get rid of the auxiliary condition "to be the identity at the boundary $\partial T_0$" imposed to $h$, by using Siebenmann's results [18].

Condition (2) now. It will be sufficient to construct homeomorphisms as in (2) from $T_0$ to $T_s$ for every $s \in [0, u]$, and take $u$ sufficiently small with respect to $\varepsilon$. First remark that for a sufficiently small $u$, the fibre $g_s^{-1}(t)$ intersects transversally the sphere $S_\varepsilon = \partial B_\varepsilon$, for all $s \in [0, u]$, and for all $t \in D_\eta(c)$. Consequently one may define a homeomorphism $h' : \partial B_\varepsilon \cap T_0 \to \partial B_\varepsilon \cap T_s$. The problem is to extend it to an homeomorphism from $T_0$ to $T_s$.

Take a Milnor ball $B_{\varepsilon'} \subset B_\varepsilon$ for $g_s$ at $(p, c)$. It appears that $(B_\varepsilon \setminus \mathring{B}_{\varepsilon'}) \cap g_s^{-1}(c)$ is diffeomorphic to $(\partial B_\varepsilon \cap g_s^{-1}(c)) \times [0, 1]$. This would be a consequence of the h-cobordism theorem (the condition $n \neq 3$ is needed here) provided that it can be applied. The argument is given by Lê-Ramanujam's in [13] and we show how this adapts to our setting. One first notices that Lê-Ramanujam's argument works as soon as one has the following conditions: for $b \in D_\eta(c)$ and $b \neq c$, the fibres $B_\varepsilon \cap g_0^{-1}(b)$ and $B_{\varepsilon'} \cap g_s^{-1}(b)$ are singular

only at $(p, b)$, they are homotopy equivalent to a bouquet of spheres $S^{n-1}$ and the number of spheres is the same. In [13] the fibres are non-singular, but non-singularity is only needed at the intersection with spheres $\partial B_\varepsilon$ and $\partial B_{\varepsilon'}$. In our setting both fibres are singular Milnor fibres of functions with isolated singularity on stratified hypersurfaces and in such a case, Lê's result [11] tells that they are, homotopically, wedges of spheres of dimension $n-1$. Now by [19, Theorem 3.1, Cor. 3.5], the number of spheres is equal to $\lambda_{p,c}(0)$ and $\lambda_{p,c}(s)$ respectively. Since $\lambda_{p,c}(s)$ is independent of $s$ by Proposition 7, these two numbers coincide.

This shows that one may apply the h-cobordism theorem and conclude that there exists a $C^\infty$ function without critical points on the manifold $(B_\varepsilon \setminus \mathring{B}_{\varepsilon'}) \cap g_s^{-1}(c)$, having as levels $\partial B_\varepsilon \cap g_s^{-1}(c)$ and $\partial B_{\varepsilon'} \cap g_s^{-1}(c)$. This function can be extended, with the same property, first on a thin tube $(B_\varepsilon \setminus \mathring{B}_{\varepsilon'}) \cap g_s^{-1}(\Delta)$, where $\Delta$ is a small enough disk centred at $c$, then further gluing to the distance function on $B_{\varepsilon'} \cap g_s^{-1}(\Delta)$. This extension plays now the role of the distance function in the construction of the contracting vector field on $T_s$. Finally, this vector field is used to extend the homeomorphism $h$ from the boundary to the interior of $T_s$, by a similar construction as the one used in proving condition (1).

The conditions (1) and (2) are now proved and therefore Lemma 10 can be applied. Its conclusion is just Theorem 9. $\square$

## 6. Proof of the main theorem

We first prove Theorem 1 in case the coefficients of the family $P$ are polynomials in the variable $s$. The general case of continuous coefficients will follow by a constructibility argument.

**6.1. Transversality in the neighbourhood of infinity.** Let $R_1 > 0$ such that for all $R \geqslant R_1$ and all $c \in \mathcal{B}_{inf}(0)$ the intersection $f_0^{-1}(c) \cap S_R$ is transversal. We choose $0 < \eta \ll 1$ such that for all $c \in \mathcal{B}_{inf}(0)$ and all $t \in D_\eta(c)$ the intersection $f_0^{-1}(t) \cap S_{R_1}$ is transversal. We set

$$K(0) = D \setminus \bigcup_{c \in \mathcal{B}_{inf}(0)} \mathring{D}_\eta(c)$$

for a sufficiently large disk $D$ of $\mathbb{C}$. There exists $R_2 \geqslant R_1$ such that for all $t \in K(0)$ and all $R \geqslant R_2$ the intersection $f_0^{-1}(t) \cap S_R$ is transversal (see [22, Prop. 2.11, Cor. 2.12] for a more general result, or see [2, Lemma 5]).

By Proposition 7, $\mathcal{B}(s)$ is an algebraic braid so we may assume that for a large enough $D$, $\mathcal{B}(s) \subset \mathring{D}$ for all $s \in [0, u]$. Moreover there exists a diffeomorphism $\chi : \mathbb{C} \times [0, u] \to \mathbb{C} \times [0, u]$ with $\chi(x, s) = (\chi_s(x), s)$ and such that $\chi_0 = \mathrm{id}$, that $\chi_s(\mathcal{B}(s)) = \mathcal{B}(0)$ and that $\chi_s$ is the identity on $\mathbb{C} \setminus \mathring{D}$, for all $s \in [0, u]$. We set $K(s) = \chi_s^{-1}(K(0))$.

We may choose $u$ sufficiently small such that for all $s \in [0, u]$, for all $c \in \mathcal{B}_{inf}(0)$ and all $t \in \chi_s^{-1}(D_\eta(c))$ the intersection $f_s^{-1}(t) \cap S_{R_1}$ is transversal. We may also suppose that for all $s \in [0, u]$, for all $t \in K(s)$ the intersection

$f_s^{-1}(t) \cap S_{R_2}$ is transversal. Notice that the intersection $f_u^{-1}(t) \cap S_R$ may not be transversal for all $R \geqslant R_2$ and $t \in K(s)$.

6.2. **Affine part.** We denote

$$B'(s) = \big(f_s^{-1}(D) \cap B_{R_1}\big) \cup \big(f_s^{-1}(K(s)) \cap B_{R_2}\big), \quad s \in [0,u].$$

By using Timourian's theorem at the affine singularities and by gluing the pieces with vector fields as done in [2, Lemma 15], we get the following trivialisation:

$$\begin{tikzcd} B' \arrow[r, "\Omega^{\mathit{aff}}"] \arrow[d, "F"'] & B'(0) \times [0,u] \arrow[d, "f_0 \times \mathrm{id}"] \\ D \times [0,u] \arrow[r, "\chi"'] & D \times [0,u], \end{tikzcd}$$

where $B' = \bigcup_{s\in[0,u]} B'(s) \times \{s\}$ and $F(x,s) = (f_s(x), s)$.

6.3. **At infinity, around an atypical value.** It remains to deal with the part at infinity $f_s^{-1}(D) \setminus \mathring{B}'(s)$ according to the decomposition of $D$ as the union of $K(s)$ and of the disks around each $c \in \mathcal{B}_{\mathit{inf}}(s)$. For each singular point $(p, c(0))$ at infinity we have a Milnor tube $T_{p,0}$ defined by a Milnor ball of radius $\varepsilon(p, c(0))$ and a disk of radius $\eta$, small enough in order to be a common value for all such points.

Let $g_s$ be the restriction to $\mathbb{X}(s)$ of the compactification $\tau$ of the $f_s$, let $G : \mathbb{X} \longrightarrow \mathbb{C} \times [0,u]$ be defined by $G(x,s) = (g_s(x), s)$ and let $C'(s) = g_s^{-1}(\chi_s^{-1}(D_\eta(c(0)))) \setminus (\mathring{B}_{R_1} \cup \bigcup_{p(s)} \mathring{T}_{p(s)})$. Now $g_s$ is transversal to the following manifolds: to $T_{p(s)} \cap \partial B_\varepsilon$, for all $s \in [0,u]$, by the definition of a Milnor tube, and to $S_{R_1} \cap C'(s)$, by the definition of $R_1$. We shall call the union of these sub-spaces the *boundary of* $C'(s)$, denoted by $\delta C'(s)$. Let us recall from §2 the definition of the Whitney stratification on $\mathbb{X}(s)$ and remark that $C'(s) \cap \operatorname{Sing} \mathbb{X}(s) = \emptyset$. Therefore $g_s$ is transversal to the stratum $C'(s) \cap \mathbb{X}^\infty(s)$.

Let $C' = \bigcup_{s\in[0,u]} C'(s) \times \{s\}$ and remark that $C' \cap \operatorname{Sing} \mathbb{X} = \emptyset$ and that the stratification $\{C' \setminus \mathbb{X}^\infty, \mathbb{X}^\infty\}$ is Whitney. Then by our assumptions and for small enough $u$, the function $G$ has maximal rank on $\bigcup_{s\in[0,u]} \mathring{C}'(s) \times \{s\}$, on its boundary $\delta C' = \bigcup_{s\in[0,u]} \delta C'(s) \times \{s\}$ and on $C' \cap \mathbb{X}^\infty(s)$. By Thom-Mather's first Isotopy Theorem, $G$ is a trivial fibration on $C' \setminus \mathbb{X}^\infty$. More precisely, one may construct a vector field on $C'$ which lifts the vector field $(\frac{\partial \chi_s}{\partial s}, 1)$ of $D \times [0,u]$ and which is tangent to the boundary $\delta C'$ and to $C' \cap \mathbb{X}^\infty$. We may in addition impose the condition that it is tangent to the sub-variety $g_s^{-1}(\partial \chi_s^{-1}(D_\eta(c(0)))) \cap S_{R_2}$. We finally get a trivialisation of $C'$, respecting fibres and compatible with $\chi$.

6.4. **Gluing trivialisations by vector fields.** Since this vector field is constructed such that to coincide at the common boundaries with the vector field defined on each tube $T$ in the proof of Theorem 9, and with the vector field on $B'$ as defined above, this enables one to glue all the resulting trivialisations over $[0,u]$. Namely, for

$$B''(s) := \left(f_s^{-1}(D) \cap B_{R_2}\right) \cup \left(f_s^{-1}(D \setminus \mathring{K}(s))\right) \text{ and } B'' := \bigcup_{s\in[0,u]} B''(s) \times \{s\}$$

we get a trivialisation:

$$\begin{array}{ccc} B'' & \xrightarrow{\Omega} & B''(0) \times [0,u] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_0\times \mathrm{id}} \\ D \times [0,u] & \xrightarrow[\chi]{} & D \times [0,u]. \end{array}$$

This diagram proves the topological equivalence of the maps $f_0 : B''(0) \longrightarrow D$ and $f_u : B''(u) \longrightarrow D$.

6.5. **Extending topological equivalences.** By the transversality of $f_0^{-1}(K(0))$ to the sphere $S_R$, for all $R \geqslant R_2$, it follows that the map $f_0 : B''(0) \longrightarrow \mathring{D}$ is topologically equivalent to $f_0 : f_0^{-1}(\mathring{D}) \longrightarrow \mathring{D}$, which in turn is topologically equivalent to $f_0 : \mathbb{C}^n \longrightarrow \mathbb{C}$.

We take back the argument for $f_0$ in §6.1 and apply it to $f_u$: there exists $R_3 \geqslant R_2$ such that $f_u^{-1}(t)$ intersects transversally $S_R$, for all $t \in K(u)$ and all $R \geqslant R_3$. Now, with arguments similar to the ones used in the proof of the classical Lê-Ramanujam theorem (see e.g. [22, Theorem 5.2] or [2, Lemma 8] for details), we show that our hypothesis of the constancy of $\mu+\lambda$ allows the application of the h-cobordism theorem on $B'''(u) \setminus \mathring{B}''(u)$, where $B'''(u) = \left(f_s^{-1}(D) \cap B_{R_3}\right) \cup \left(f_s^{-1}(D \setminus \mathring{K}(s))\right)$. Consequently, we get a topological equivalence between $f_u : B''(u) \longrightarrow D$ and $f_u : B'''(u) \longrightarrow D$. Finally $f_u : B'''(u) \longrightarrow \mathring{D}$ is topologically equivalent to $f_u : f_u^{-1}(\mathring{D}) \longrightarrow \mathring{D}$ by the transversality evoked above, and this is in turn topologically equivalent to $f_u : \mathbb{C}^n \longrightarrow \mathbb{C}$.

6.6. **Continuity of the coefficients.** So far we have proved Theorem 1 under the hypothesis that the coefficients of the family $P$ are polynomials in the parameter $s$. We show in the last part of the proof how to recover the case of continuous coefficients. The following argument was suggested to the first named author by Frank Loray. Let $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$ be the set of polynomials of degree $d$, with isolated singularities in the affine space and at infinity, with fixed number of vanishing cycles $\mu+\lambda$ and with a fixed number of atypical values $\#\mathcal{B}$. Proposition 5 tells that $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$ is a constructible set. Since $f_0$ and $f_1$ are in the same connected component of $\mathcal{P}_d(\mu+\lambda, \#\mathcal{B})$, we may connect $f_0$ to $f_1$ by a family $g_s$ with $g_0 = f_0$ and $g_1 = f_1$ such that the coefficients of $g_s$ are piecewise polynomial functions in the variable $s$. Using the proof done before for each polynomial piece, we

finally get that $f_0$ and $f_1$ are topologically equivalent. This completes the proof of Theorem 1.

## References


[1] N. A'Campo, Le nombre de Lefschetz d'une monodromie, Indag. Math. 35 (1973), 113–118.

[2] A. Bodin, Invariance of Milnor numbers and topology of complex polynomials, Comment. Math. Helv. 78 (2003), 134–152.

[3] A. Bodin, Non-reality and non-connectivity of complex polynomials, C. R. Math. Acad. Sci. Paris 335 (2002), 1039–1042.

[4] S.A. Broughton, Milnor numbers and the topology of polynomials hypersurfaces, Inv. Math. 92 (1988), 217–241.

[5] D. Burghelea, A. Verona, Local homological properties of analytic sets, Manuscripta Math. 7 (1972), 55–66.

[6] D. Eisenbud, Commutative algebra, Graduate texts in mathematics 150, Springer-Verlag, 1994.

[7] L. Fourrier, Topologie d'un polynôme de deux variables complexes au voisinage de l'infini, Ann. Inst. Fourier 46 (1996), 645–687.

[8] T. Fukuda, Types topologiques des polynômes, Inst. Hautes Études Sci. Publ. Math. 46 (1976), 87–10.

[9] H.C. King, Topological type in families of germs, Invent. Math. 62 (1980/81), 1–13.

[10] F. Lazzeri, A theorem on the monodromy of isolated singularities, Singularités à Cargèse, Astérisque 7-8 (1973), 269–275.

[11] Lê D.T., Complex analytic functions with isolated singularities, J. Algebraic Geom. 1 (1992), 83–99.

[12] Lê D.T., Une application d'un théorème d'A'Campo à l'équisingularité, Nederl. Akad. Wetensch. Proc. (Indag. Math.) 35 (1973), 403–409.

[13] Lê D.T., C.P. Ramanujam, The invariance of Milnor's number implies the invariance of the topological type, Amer. J. Math. 98 (1976), 67–78.

[14] Lê D.T., C. Weber, A geometrical approach to the Jacobian conjecture for $n = 2$, Kodai Math. J. 17 (1994), 374–381.

[15] A. Libgober, Homotopy groups of the complements to singular hypersurfaces, II, Ann. of Math. 139 (1994), 117–144.

[16] A. Libgober, M. Tibăr, Homotopy groups of complements and nonisolated singularities, Int. Math. Res. Not. 2002, no. 17, 871–888.

[17] J. Milnor, Singular points of complex hypersurfaces, Ann. of Math. Studies 61, Princeton University Press 1968.

[18] L.C. Siebenmann, Deformation of homeomorphisms on stratified sets, I, II, Comment. Math. Helv. 47 (1972), 123–163.

[19] D. Siersma, M. Tibăr, Singularities at infinity and their vanishing cycles, Duke Math. J. 80 (1995), 771–783.

[20] D. Siersma, M. Tibăr, Deformations of polynomials, boundary singularities and monodromy, Moscow Math. J. 3 (2003), 661–679.

[21] R. Thom, Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc. 75 (1969), 249–312.

[22] M. Tibăr, Regularity at infinity of real and complex polynomial functions. Singularity theory (Liverpool, 1996), London Math. Soc. Lecture Note Ser. 263, pp. 249–264, Cambridge Univ. Press 1999.

[23] J.G. Timourian, The invariance of Milnor's number implies topological triviality, Amer. J. Math. 99 (1977), 437–446.

*E-mail address*: Arnaud.Bodin@math.univ-lille1.fr

*E-mail address*: Mihai.Tibar@math.univ-lille1.fr

Laboratoire Paul Painlevé, Mathématiques, Université Lille I, 59655 Villeneuve d'Ascq, France.